Linear Transformations Linear Algebra X. Du

- Suppose you want to transform a vector from one thing to another. Examples:
  - Rotating a vector to rotate an image.
  - Shearing a vector to distort an image.
  - Reflecting a vector to reflect an image.
  - Project a 3D vector onto a 2D plane.
    - This is very useful in computer imagery and graphics.
- A linear transformation can be represented as $T(\vec{x}) = A\vec{x}$, where A is the transformation matrix.
- If T is a transformation from n-space to m-space, then A will be an $m \times n$ matrix.
- Common transformation matrices in 2-space:
  - Reflection across the y-axis: $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
  - Projection onto the x-axis: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
  - Rotation counterclockwise: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
  - Dilation: $A = kI$
- Suppose $T(\vec{x}) = T_2(T_1(\vec{x}))$, or $T = T_2 \circ T_1$, then its transformation matrix will be $A = A_2 A_1$
- Suppose A is an $n \times n$ matrix, and $T(\vec{x}) = A\vec{x}$, where $\vec{x}$ lives in n-space. Then T is a bijection iff A is invertible.
  - $T(\vec{x}) = A\vec{x} \Leftrightarrow T^{-1}(\vec{x}) = A^{-1}\vec{x}$
- Axioms of linear transformations $T: V \to W$
  - $T(\vec{0}_V) = \vec{0}_W$
  - $T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$
  - $T(c\vec{a}) = cT(\vec{a})$
- **Kernel**: $\ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}$
  - If $T(\vec{x}) = A\vec{x}$, then $\ker(T) = \text{null}(A)$
  - The kernel is a subspace of V.
  - T is a bijection iff $\ker(T) = \{\vec{0}\}$
- **Rank-Nullity Theorem**
  - For a linear transformation, rank(T) = dim(range(T)), and null(T) = dim(ker(T))
  - For matrices, rank(A) + null(A) = n, where n is the number of columns of A
  - For linear transformations: dim(range(T)) + dim(ker(T)) = dim(V)
    - Also written as rank(T) + null(T) = dim(V)